INSTRUCTORS' FACILITATION OF STUDENT PARTICIPATION IN ADVANCED MATHEMATICS LECTURES: A CASE STUDY OF TWO INSTRUCTORS

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Literature portrays advanced mathematics lecturing as a uniform teaching style, often criticized for offering minimal opportunities for student participation. In this paper we present results from a comparative case study of two instructors' facilitation of student participation in Real Analysis lectures. Analyzing fieldnotes from several observed lectures of each instructor, we found that the two instructors' facilitation of student participation during lectures consistently differed in (1) the participation structures were used, (2) the types of questions asked, and (3) how instruction responded to students' contributions. Our findings show that lecturing in advanced mathematics is not a uniform style and that active student participation in lectures is possible. We interpret the potential impact of observed differences on students' learning and experiences in terms of the Teaching for Robust Understanding (TRU) framework (Schoenfeld, 2018).

Keywords: Instructional Activities & Practices, Undergraduate Education, Classroom Discourse

The undergraduate math education literature tends to portray lecturing in advanced mathematics as a single teaching style, in which instructors engage in "chalk talk" (writing formal mathematics on the blackboard while providing oral commentary) and student participation is minimal (Artemeva & Fox, 2011; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016; Paoletti et al., 2018). Here, we join other recent scholarship (e.g. Pinto, 2019; Viirman, 2015) in problematizing this homogenous picture of advanced mathematics lectures. We report on variability in two instructors' practices of facilitating student participation during Real Analysis lectures, discussing potential implications of each approach for student experience in terms of the Teaching for Robust Understanding (TRU) framework, which delineates five dimensions of classroom practice important for learning (Schoenfeld, 2018).

Active student participation leads to more robust learning outcomes than passive observation (Chi & Wylie, 2014), specifically in the context of tertiary STEM education (Freeman et al., 2014). Furthermore, agentic participation in classroom discourse provides students with opportunities to develop productive disciplinary dispositions and identities (Gresalfi, Martin, Hand, & Greeno, 2009). Because of this and based on the shared assumption that lectures involve minimal student engagement, many mathematics education researchers argue against lecturing, and instead advocate for student centered teaching approaches such as Inquiry Based Learning (IBL: Laursen & Rasmussen, 2019). Yet others argue that it is neither realistic nor desirable to abandon lectures altogether (Sfard, 2014). Most advanced mathematics courses are still taught in a lecture format (Johnson, Keller, & Fukawa-Connelly, 2018), and mathematicians – many of whom claim that lectures are "the best way to teach" – are not likely to abandon lecturing in favor of radically different teaching approaches (Woods & Weber, 2020).

Instead of a complete overhaul of lecturing, several scholars suggest "tilting the classroom" (Alcock, 2018) – incorporating minimally invasive active-learning strategies into traditional lecturing (Woods & Weber, 2020). Suggestions include using participation routines such as Think-Pair-Share and classroom polls (Braun, Bremser, Duval, Lockwood, & White, 2019). However, there is little research on how implementation of such strategies actually looks like in

the context of advanced mathematics lectures (Woods & Weber, 2020), and what such practices may be good *for*. Our paper begins to address this gap in that it provides an analysis of two distinct examples of how instructors facilitate active student participation during lectures and discusses the relative affordances of each for student experience and learning.

Participation in Classroom Practices and the TRU framework

We draw on sociocultural theories that conceptualize disciplinary knowledge as participation in disciplinary practices (Lave, 1996). Within this tradition, participation in classroom practices is seen as a central mechanism by which students both learn disciplinary content and develop disciplinary identities and dispositions (Cobb, Stephan, McClain, & Gravemeijer, 2001; Esmonde, 2009). Hence, an important aspect of teaching is how instructors organize classroom environments to support student participation, including the specific moves they make to invite and facilitate such participation. What matters is not just *that* students participate, but also *what* they get to participate in, and *how*; the nature of classroom practices shapes the kinds of content, skills, and dispositions students develop through their participation (Gresalfi et al., 2009).

The Teaching for Robust Understanding (TRU) framework is a synthesis of such research on aspects of classroom practice associated with robust learning outcomes (Schoenfeld, 2018). The framework delineates five dimensions – *(1) the content, (2) cognitive demand, (3) equitable access, (4) agency, ownership and identity, and (5) formative assessment* – meant to comprehensively characterize a classroom environment. *The content* captures the extent to which the enacted mathematics is coherent, connected and centered on important practices and ideas. *Cognitive demand* refers to the degree to which classroom activities provide opportunities for "productive struggle", hitting the sweet spot of accessibility and challenge. *Equitable access* refers to classroom practices that ensure that *all* students have opportunities for meaningful engagement with disciplinary activities. *Agency, Ownership, and Identity* refers to the extent to which students get to make significant choices, generate mathematical content themselves, and get positioned as mathematically competent by having their ideas built on in the classroom. *Formative assessment* refers to the extent to which student thinking – including productive beginnings and possible misunderstandings – is surfaced and responded to in instruction.

In this paper, we use TRU to organize our findings about instructors' practices of facilitating student participation in ways that directly connect them to central mechanisms of learning in classrooms. We focus on dimensions 2-5 of TRU, for which student participation is relevant. Our focal research question is: *how did the observed lecturers facilitate student participation, and what significance might these moves have for student learning and experience?*

Methods

Context & Data Collection

The data our project is grounded in are observational fieldnotes of Real Analysis lectures taught in a large research university in the western U.S. during Fall 2019. We observed three instructors teaching the same elective upper division undergraduate course in parallel during a period of 3 weeks in the first half of the semester. Due to space limitations, here we focus only on Dr. A and Dr. B. We observed eight 50-minute lectures taught by Dr. A and four 80-minute lectures taught by Dr. B (a total of 400 and 320 minutes, respectively). Each lecture-section was attended by approximately 30 students.

Each author took detailed fieldnotes using a note-taking instrument specifically designed for our analytic purposes. The top third of each page was used to document blackboard inscriptions

verbatim. The bottom two-thirds were devoted to descriptions of participants' speech and actions. This section was further divided into two columns: the left column for general actions made by the instructor, the right column for events related to student participation. Board inscriptions and speech/action descriptions were linked to one another by number indexing, allowing us to reconstruct the sequential organization of each lecture, coordinate speech and inscriptions, and systematically document student participation – the analytic focus of this paper.

Data Analysis

Our analysis proceeded both inductively and deductively. Each author reviewed their own fieldnotes independently to identify participation facilitation routines. In subsequent analytic discussions, we consolidated our independent characterizations, and organized them into three aspects of facilitation practice: what participation-structures were used, what kinds of questions instructors asked, and how instructors responded to student contributions. Next, we took a deductive approach. We turned to the literature to learn about how these three aspects (participation structure, teacher questions, response to students) were defined in prior work. This process helped us refine and operationalize our initial analytic categories. With refined definitions, we returned to the data, and coded all instances of student participation in relation to these categories. What follows is an explanation of the analytic categories we used.

Participation structures are defined as "the organization of persons' reciprocal rights and obligations in social interaction" (Erickson & Mohatt, 1977, p. 139). Common classroom structures are: whole-class discussion, group work, and individual work. One can, however, further specify "persons' reciprocal rights and obligations" within each broad category. Thus, in addition to a general *participation structure* category, we looked at *initiation type*, where type refers to who has the right and obligation to respond to a particular initiation (some student, a specific student, all students), and *who participated* as a result (all or some students).

In the literature, teacher questions are typically coded for the kind of responses they require from students. While precise category-labels vary, most studies use a coding scheme that organizes question-types in an hierarchical progression, from low- to high-order questions (DeJarnette, Wilke, & Hord, 2020). We adopted a similar approach, and categorized instructors' mathematical questions broadly as either *low, medium or high level* in terms of the question's cognitive demand (e.g. definition recall vs. proof idea) and openness (a single correct answer vs. many possible answers). In addition to mathematical questions and tasks, we also coded instances of: non-mathematical questions, solicitations of questions from students (e.g. "do you have any questions?") and comprehension-monitoring questions (e.g. "does that make sense?").

Finally, to characterize responses to student contributions, we first noted the *extent* (in terms of length and complexity) of student contributions. We then noted whether instructors' initial *response to student contributions* was encouraging or evaluative, as well as the extent to which subsequent lecture-talk *builds on student ideas*. Finally, we noted whether response episodes involved a single exchange sequence (a student asks a question, the instructor answered) or several back-and-forth exchanges, as these are indicative of the extent to which classroom discourse is dialogic and ideas are co-constructed (Wells & Mejia Arauz, 2006).

The last step of our analysis involved connecting these observables to the dimensions of the TRU framework. The same behavioral indicator can be implicated in more than one TRU dimension. For example, a student's opportunity to explain their thinking in relation to a conceptually rich task involves *cognitive demand,* disciplinary *agency*, and constitutes a *formative assessment* opportunity for the teacher. Thus, for each dimension of TRU, we used the

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definition of the dimension to read across the coding described above and synthesized all aspects of instructors' facilitation of student participation that contribute to that dimension.

Findings & Discussion

The table below summarizes our findings. The two sections that follow provide detailed descriptions and examples of each instructor's facilitation practices, unpacking the codes and descriptions in the table. In the last findings section, we interpret the potential impact of identified practices in terms of the four selected TRU dimensions: *(2) cognitive demand, (3) equitable access, (4) agency, ownership and identity, and (5) formative assessment*.

Table 1: Facilitation of Student Participation - Dr. A and Dr. B

	Dr. A		Dr. B
Participation Structure	Whole-Class	Think-Pair-Share	Whole-Class
Initiation Type	Voluntary Q & A Poll	Pair: *"Discuss with your neighbor..."* Share: randon calling using name-cards	Voluntary Q & A Poll
Who participated?	A few students	All students	A few students
Types of Tasks	Math: low-level	Math: med-level	Math: low, med & high-level
Types of Questions	Open solicitation: *"What are your questions so far?"*	Open solicitation: *"X, what are your thoughts, ideas, and questions here?"*	Non-mathematical, Comprehension monitoring: *"Does that make sense?"*, Open solicitation: *"Any questions here?"*
Length of student contributions	Short and quick take-over		Some short & some extended
Instructor's response to student contributions	Encouraging (all contribution types): *"That's a great question!"*		Endorsing (correct answers): *"You got it!"* ; *"Perfect!"*
Lecture building on student ideas	Short		Some short & some extended
Extent of dialogue	Single exchange		Some single exchange some back-and-forth

Facilitation of Student Participation:

Dr. A. Dr. A utilized two distinct participation structures in his lectures: a traditional whole-class format and a participation structure known as "Think-Pair-Share" (Braun et al., 2019).

During whole-class (i.e., outside of Think-Pair-Share), Dr. A invited students to participate by asking questions addressed to all students, to which individual students could respond verbally on voluntary basis. Dr. A rarely asked "known-answer-questions" in the whole-class format. The vast majority of questions Dr. A posed were open solicitation questions such as "what do you notice?" or "what are your comments so far?". *Any* thematically relevant thoughts, comments or questions could count as a legitimate student response, and indeed, students utilized these open-solicitation prompts as invitations to ask questions and clarify confusions. Whenever Dr. A asked specifically mathematical question during whole-class, the question required only short responses from students. Several times during our observation period, Dr. A initiated polls; prompts involving several specified answer-options (e.g., "should this be $\leq$, $\geq$, or $=$?") on which *all* students were asked to vote, though often only a subset of them did. Occasionally, Dr. A initiated short IRE exchanges in the middle of an explanation (e.g. "what is $cos\ cos\ \pi/3?$"). The purpose seemed to be ensuring that all students are "on the same page".

Think-Pair-Share was a salient participation structure in Dr. A's class. Several times each lecture (at least 2-3 times), Dr. A introduced a short mathematical task and asked students to discuss it with their "neighbor." Students were seemingly used to this routine by the time of our

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observations; all seemed to work on the task and lively talk could be heard. Meanwhile, Dr. A left his front-of-the-room position, circulated among the desks, and stopped to talk with students (typically called-on by the students themselves). After a few minutes, Dr. A called the class back to a whole-class format and nominated a specific student to "share." The choice of student to call on was done through a randomized selection mechanism: a stack of cards with students' names. Students cooperated with the "cold-calling" approach, and to the extent evident from our observations, seemed to be comfortable with it. In the public "share" phase, Dr. A initiated student contributions with open-solicitation prompts such as "what are your thoughts, ideas, questions here?" The format implied that it was acceptable to "share" by stating a confusion, a hesitation or asking a question rather than providing a direct answer to the original task. Indeed, students often responded by asking a question and sometimes by saying "I don't know".

The tasks Dr. A used for pair activities incorporated an assortment of mathematical practices. To name a few examples, students were asked to: determine if an example satisfies a definition, complete a few steps in a proof, interpret a theorem-statement in terms of a diagram, and complete a proposition claim. One overarching characteristic of such tasks is that they are all at a medium-level in terms of task-openness and cognitive demand; they involve non-trivial mathematical engagement from the students, yet were structured and accessible. A similar approach to selecting classroom activities for Real Analysis lectures was articulated by Alcock (2018), where she described such activities as "short-and-snappy" and geared toward "conceptual understanding" rather than calculations (p. 24).

Whether in the whole-class format or the "share" phase of Think-Pair-Share, students' verbal contributions in the public sphere of the classroom were short. In whole class, Dr. A's initiations typically called-for short responses only. In the "share" part of pair-share, students sometimes began articulating what could potentially be a longer contribution, but in all of our observations, Dr. A quickly took over. The impression we had was that Dr. A took over to make the explanation clearer to the whole class and cut students' own explanations short to "save time". In general, there was a sense of fast pace in Dr. A's class, both in terms of the speed of Dr. A's "chalk-talk" and because of swift transitions between many planned activities.

Whenever a student asked a question or offered an idea, Dr. A responded encouragingly (e.g. "great question!"), with level of enthusiasm independent of the contribution's sophistication or correctness. Whenever a student's question generated new mathematical content, Dr. A took it up and responded to it (e.g., by answering the question). However, in all of our observations, such contributions never led to an extended whole-class discussion or significantly altered the lecture. Furthermore, they rarely involved a back-and-forth of ideas between Dr. A and the student, or other students. For example, once a student asked why two quantities are equal. Answering that question prompted Dr. A to write a sketch of a proof for "why it should be equal" on the board. Thus, the student's question altered the course of the lecture slightly, and prompted Dr. A to generate new content, both verbally and in writing. However, the episode was short and neither the student who initiated the question nor other students made any follow-up contributions.

Dr B. Dr. B's lecturing was "traditional" in that it was conducted through a whole-class participation structure, "chalk-talk" was a pervasive discourse genre, and student participation was organized primarily through voluntary question-answer exchanges with the instructor.

A salient feature of Dr. B's lecturing was frequent use of short IRE sequences. This involved both low-level math tasks, such as simple recall, and tasks we considered to be medium-level, such as discerning proof-structure. A common routine in Dr. B's class featuring low-level IRE

sequences can be broadly described as scaffolded "chalk-talk". Dr. B often prompted students to verbalize the precise formulation of a definition, and then used students' responses to "dictate" her board writing. As a mathematical task, this routine is closed and involves low demand. Yet, given that verbalizing formal mathematical texts can be difficult for newcomers (Shepherd & van de Sande, 2014), such a scaffolded version of "chalk-talk" (in which the instructor does the "chalk", and a student does the "talk") could have important benefits. A medium-level IRE routine Dr. B often used was to engage students in discerning logical-structure. She prompted students to recognize assumptions, or givens, by asking questions such as "What else do I know?" or "What is my claim?". While such questions follow a relatively narrow path (a student responded, Dr. B endorsed it), discerning proof-structure is a non-trivial skill for students to learn in proof-based courses (Selden & Selden, 1995). Thus, we considered such tasks to be at a medium-level, as they required actions that went beyond recall and recitation.

Though less frequently, Dr. B also posed questions that can be considered high-level in terms of openness and cognitive demand. On several occasions, Dr. B invited students to suggest mathematical examples, contribute central steps in an argument, or generate proof ideas. Occasionally, Dr. B made use of classroom voting, or polls. This seemed to primarily function as a formative assessment mechanism, that is, as means for Dr. B to gauge whether all students are following and could discern the correct option. Dr. B also routinely asked short comprehension monitoring (e.g. "does that make sense?") and open-solicitation (e.g. "any questions?) questions. Students rarely volunteered verbal responses to such questions, though we presume that Dr. B read some non-verbal feedback from students. Notably, Dr. B also regularly asked questions engaging students in decisions pertaining to mathematical conventions such as notation (e.g. "what letter should I used?"), as well as non-mathematical questions (e.g. "what's that French word for combining two words together?"). Such questions invited broad student participation.

Students' mathematical contributions, whether as responses to Dr. B's initiations or initiated by the students themselves, varied in extent and complexity, ranging from single phrase responses to lengthy articulations of mathematical scenarios and ideas. Furthermore, extended student contributions were often situated within longer classroom episodes that involved substantial building on student ideas and several back-and-forth exchanges between students and Dr. B. The following vignette serves as an example of one such case:

> Dr. B introduced a definition for *limit points* of a set in a metric space and offered an illustrative context: R (the set of real numbers) as the metric space, the interval $S = [0,1]$ as a set under consideration, and the 1 as a possible *limit point* of $S = [0,1]$. Later, she wrote the following statement on the blackboard: $1 \in lim\ (S) \Leftrightarrow \exists (x_n)$ in S s.t. $x_n \to 1$ as $n \to \infty$, which can be read as: the number 1 is a limit of point S if and only if there exists a sequence of numbers all of which are within the interval $[0,1]$ such that this sequence approaches 1 at infinity. Referring to the sequence x_n, Dr. B asked students "What's an example?", and a student suggested the sequence $x_n = 1 - \frac{1}{n}$. Dr. B picked up and elaborated on this suggestion by writing several elements in this sequence, and indicating that each belongs to the interval S: $x_1 = 0 \in S, x_2 = \frac{1}{2} \in S, x_3 = \frac{2}{3} \in S \quad x_4 = \frac{3}{4} \in S \ldots x_n = 1 - \frac{1}{n} \in S$. The last equality refers to the general pattern: *any* element in the sequence x_n belongs to the interval $S = [0,1]$. She then turned to the class and asked, "how do we prove it?". Another student (not the same one that originally suggested the sequence), provided an elaborate answer

constituting steps of a proof. Dr. B picked-up and "re-voiced" the student's idea by producing a short proof-text mirroring the argument the student described verbally.

This episode illustrates several of the general trends mentioned above. High-level mathematical questions from Dr. B, while not the most frequent form of questions used in her class, prompted students to contribute ideas, both short (e.g. suggesting an example sequence "$x_n = 1 - \frac{1}{n}$") and extended (e.g. verbally describing a proof-argument). When ideas were suggested, Dr. B took them up by writing them on the board, and further elaborated and explained them. Dr. B often posed further questions in the context of an initial student-suggestion. In this case, the question "how do we prove it?" prompted another student's response. Such moves initiated extended episodes that featured back-and-forth exchanges between instructor and students, and at times, allowed students to build on one another's ideas.

Interpretation of instructors' facilitation of student participation using TRU

Cognitive Demand. In Dr. A's class, Pair-Share activities provided students with routine opportunities to engage in mathematical tasks. The tasks Dr. A used were at a consistent medium-level of demand; they involved non-trivial mathematical engagement such as interpreting a proposition statement, or completing steps in a proof, yet were concrete and accessible. In Dr. B's class, there was greater variability in terms of tasks' cognitive demand. Dr. B's questions ranged from basic recall and verbalization questions (e.g., dictate a definition), to medium-level proposition-structure tasks (e.g., "what is my claim?"), and up to open and cognitively demanding questions (e.g., "how do we prove this?"). Both approaches – Dr. A's keeping cognitive demand at a consistently moderate level, and Dr. B's varying demand level from low to high within a single lecture – provided "productive struggle" opportunities.

Equitable Access. Several of Dr. A's teaching routine supported *equitable access* to mathematical content and practices. His frequent use of the Pair-Share participation structure provided opportunities for *all* students to actively engage with non-trivial mathematical tasks. In addition, the open-format of the questions Dr. A posed ("what are your thoughts?"), and his explicitly affirmative responses ("that's a great question!"), reduced access barriers for students' verbal participation during whole class, since all types of contributions (whether questions, suggestions or articulated confusion) were framed as legitimate and valuable. In Dr. B's class, we identified few explicit mechanisms that supported equitable access to mathematical content. Dr. B asked questions frequently, and several students participated verbally, but this participation was not distributed equally and the experiences of students who did not participate verbally is difficult to gauge (though we do not assume silent students did not participate, see e.g. (O'Connor, Michaels, Chapin, & Harbaugh, 2017)). Dr. B's non-mathematical questions seemed to encourage broader participation. However, her closed-form mathematical questions were typically responded to by a single student (once a correct response was given, no further were needed). Most notably, extended discussions (as describe above) seemed to engage a select few.

Agency, Ownership, Identity (AOI). In Dr. A's class, it was not clear to what extent students had opportunities to see themselves and peers as creators of mathematics. The selected tasks supported students' engagement with important mathematical practices. However, given that they were relatively narrowly circumscribed, we might ask to what extent engagement with the tasks allowed students to exercise mathematical *agency* and take *ownership* of the content. For example, a fill-in the blank task to complete a proposition text engages students in important mathematics. However, does it make students feel *ownership* over the resulting mathematical text and ideas? Similar questions can be raised about students' participation in whole class too.

Given that students' verbal contributions in whole class were short and not significantly built upon, it is not clear to what extent participation in Dr. A's whole class discussions provided students with opportunities to feel *ownership* of the content, exercise mathematical *agency*, and be positioned as competent doers and creators of mathematics. In Dr. B's class, extended episodes provided participating students with ample opportunities to exercise *agency*, feel *ownership* of ideas and be recognized as mathematically competent. Yet, as described above, only a few students in Dr. B's class had this experience. Silent observers of extended dialogic interactions are afforded valuable learning opportunities (O'Connor et al., 2017). However, the impact on students' negotiated *identities* may be more problematic. As observers in Dr. B's class, we could "read" a mathematical hierarchy among students. In contrast, in Dr. A's class, we could not discern a similar pattern: it was not easy to tell who "the smart students" were.

Formative Assessment. Dr. A's practices afforded ample opportunities to surface and notice student thinking, at all levels of correctness and completion. When walking around during pair work, Dr. A could observe and respond to students' ideas. Also, Dr. A's explicit framing of all contribution types as legitimate through open solicitations and encouragement, ensured that students voiced confusions and partial understandings, not just confident correct answers. Dr. B's facilitation afforded less systematic surfacing of student ideas. Dr. B asked many questions, yet most were responded to by one student at a time, so a range and variety of student ideas were not easy to pick up on. Importantly, students rarely voiced incorrect ideas, and mistakes and confusions were not part of the whole-class discussion in Dr. B's class.

Conclusions

Our results indicate that the observed instructors used distinct approaches to facilitate student participation. Dr. A and Dr. B utilized different participation structures, posed different kinds of questions, and used contrasting approaches in responding to students' verbal contributions. Similarly to Pinto (2019) and Viirman (2015) we found that lecturing is not a uniform teaching style. This paper contribute to the field's understanding of the nuances of mathematics teaching practices at the university level (Speer, Smith, & Horvath, 2010). However, how widespread the observed facilitation moves are among mathematics instructors remains an open question.

By interpreting observed variations using the TRU framework (Schoenfeld, 2018), we further suggested that Dr. A's and Dr. B's approaches to facilitating student participation have different consequences for learning. Dr. A's frequent use of Pair-Share activities and non-evaluative questioning routines, afforded *all* students with consistently moderate level of cognitive demand and provided ample opportunities to surface and respond to partial understandings. Students could not hide in Dr. A's class; everyone's name was called-on at some point. Yet, no student "out-shined" others; identities of competence (Gresalfi et al., 2009) were not a salient aspect of the classroom stage. Content learning opportunities were distributed more equally, but we suspect classroom interactions may have not supported students in developing strong disciplinary identities. Dr. B's diverse questions provided students with a range of "productive struggle" opportunities and contributed to the enactment of multifaceted mathematical practice. Extended, student-centered episodes provided a few students with opportunities to generate and refine mathematical ideas on a public stage in ways that positioned them as mathematically competent. For observers, this was an "existence proof" that mathematics *can* be generated by a student, and thus constituted an important socio-mathematical norm (Yackel & Cobb, 1996). But, did all students emerge to see themselves as mathematically competent? We suspect not. The extent to which each approach to facilitating

student participation (Dr. A's or Dr. B's) actually contributed to longer term outcomes for learning and identity remains an open question for future research. Coordinating such analyses with student assessment and interview data could be a productive direction for future research.

When debating the effectiveness of different teaching approaches, it is important to keep in mind there might not be a single "best" way. Here, we showed that distinct approaches to incorporate "minimally invasive" active learning in lectures are possible and may have different affordances in terms of learning and identity. Thus, negotiating our values and ultimate goals for advanced mathematics education is an important part of the conversation about what is "best".

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